Completing the Square

Everyone knows how to multiply out something like this: $(x+3)^2$. (*Please* don't say that's equal to x^2+9!!!)

You could even expand something like $2(x+3)^2+7$, if you really wanted to:

$$\begin{aligned}2(x+3)^2+7 &= 2(x+3)(x+3)+7\\ &= 2(x^2+3x+3x+9)+7\\ &= 2(x^2+6x+9)+7\\ &= 2x^2+12x+18+7\\ &= 2x^2+12x+25\end{aligned}$$

Notice what we just did. We took something that's of the form $p(x+q)^2+r$ and turned it into something of the form ax^2+bx+c. Not hard, right? Completing the square is just doing that backwards. When we complete the square, we want to take something of the form ax^2+bx+c and rewrite it in the form $p(x+q)^2+r$. With a little practice and some tricks, it's not hard to do.

Why Complete the Square?

Completing the square is just rewriting a quadratic expression in a different form. The point is that the different forms are good for different things. If I want to solve a quadratic equation, I like the form ax^2+bx+c, because then I can factor it or use the quadratic formula. If I want to know the vertex or graph a parabola, I like the form $p(x+q)^2+r$, because then I can read off the vertex and graph it using graph transformation rules. It just depends what you're trying to do.

First, let's go slow motion. We'll go through another example in the "forwards" direction, and then do the same example in the "backwards" direction to figure out some tricks. To make it easier to start, we'll have a leading coefficient of 1.

Our example is $(x+5)^2+4$. Let's multiply it out:

$$\begin{aligned}(x+5)^2+4 &= (x+5)(x+5)+4\\ &= x^2+5x+5x+25+4\\ &= x^2+10x+25+4\\ &= x^2+10x+29\end{aligned}$$

Now let's pretend that we're given the expression $x^2+10x+29$ and asked to rewrite that in the form $(x+q)^2+r$. What would we do? We can gain some clues by looking at the "forward" direction.

Notice that the $10x$ came from adding $5x+5x$, which in turn came because we had an $(x+5)^2$. This means that the coefficient 10 comes from being two times the "q" term. So if we're going backwards, we can just look at that 10, divide it by 2, and know what q should be. So at this point we could say:

$$\begin{aligned}x^2+10x+29 &= \,?\\ &= \vdots\\ &= (x+5)^2+\,?\end{aligned}$$

There are still some missing steps in between. However, the next thing we can notice is that if we want to end up with $(x+5)^2$, then we should have $x^2+10x+25$ at some point (since $(x+5)^2=x^2+10x+25$ when it's expanded). Let's fill that in:

$$
\begin{aligned}
x^2 + 10x + 29 &= ? \\
&= \vdots \\
&= x^2 + 10x + 25 + ? \\
&= (x+5)^2 + ?
\end{aligned}
$$

Now that first expression, $x^2 + 10x + 29$, we want to be equal to $x^2 + 10x + 25+?$. So what should the ? be? Clearly, it's gotta be 4. So then we can fill in the rest, and it turns out our "r" should be 4.

$$
\begin{aligned}
x^2 + 10x + 29 &= x^2 + 10x + 25 + 4 \\
&= (x+5)^2 + 4
\end{aligned}
$$

That was some good deductive reasoning to work backwards. We can make it more algorithmic as we go on, but first let's do another example, this time without knowing the answer beforehand. Let's say we start with $x^2 + 6x - 18$ and want to rewrite it in the form $(x+q)^2 + r$. Let's think it through:

(Let's see...we know that the "q" should be half of the "b," so we'll have something involving an $(x+3)^2$*...)*

(If we have $(x+3)^2$*, that's the same as* $x^2 + 6x + 9$*....but we have* $x^2 + 6x - 18$*...)*

(Ah, but $x^2 + 6x - 18$ *is the same as* $x^2 + 6x + 9 - 27$*, if we really wanted to write it that way...that's it!)*

$$
\begin{aligned}
x^2 + 6x - 18 &= x^2 + 6x + 9 - 27 \\
&= (x+3)^2 - 27
\end{aligned}
$$

The most common question to have at this point is: "Yeah, but how'd you know to write it as $x^2 + 6x + 9 - 27$?" Here's where we can introduce another trick of the trade to make that step easier. In the previous example, once we knew we wanted $(x+3)^2$, we knew that we wanted x^2+6x+9 to appear in our expression, but what we had was $x^2 + 6x - 18$. One way to make $x^2 + 6x + 9$ "appear" is to *add 9 and subtract 9* all at once. By adding and then subtracting 9, we're not actually changing anything (it's like adding 0), so it's a legal move. And it makes it easier to rewrite in the form we need. Let's see this in action:

(I'm starting with $x^2 + 6x - 18$*. I know I want* $(x+3)^2$*, which means I want* $x^2 + 6x + 9$*...)*

(Let's add and subtract 9 in $x^2 + 6x - 18$*, to get* $x^2 + 6x + 9 - 9 - 18$*.)*

(Now the first part, $x^2 + 6x + 9$*, is what I wanted to have. That equals* $(x+3)^2$*.)*

(And the second part is $-9 - 18$*, which we can combine to get* -27*. Perfect! It's* $(x+3)^2 - 27$*.)*

So now you have a few tricks to use in completing the square. Before you start practicing those, here's one more example using both tricks. This example also introduces some negative terms and fractions, so it may look more complicated...but it's the same procedure. Let's say we want to rewrite $x^2 - 5x - 3$ in the form $(x+q)^2 + r$.

Half of -5 is $-\frac{5}{2}$. So our q is $-\frac{5}{2}$, which means we want to have $(x - \frac{5}{2})^2$, which equals $x^2 - 5x + \frac{25}{4}$. To get that in our expression, we'll add and then subtract $\frac{25}{4}$, and simplify:

$$\begin{aligned} x^2 - 5x - 3 &= x^2 - 5x + \frac{25}{4} - \frac{25}{4} - 3 \\ &= (x - \frac{5}{2})^2 - \frac{25}{4} - 3 \\ &= (x - \frac{5}{2})^2 - \frac{37}{4} \end{aligned}$$

And we're done. That's how you complete the square. Now you practice:

(1) $x^2 + 12x - 10$

(2) $x^2 - 8x + 14$

(3) $x^2 + x - 1$

(4) $x^2 - 100x + 100$

(5) $x^2 + 7x + \frac{1}{2}$

(6) $x^2 - \frac{x}{3} + 1$

Many students at this point have trouble not because of the procedure for completing the square, but because they're rusty on fractions. Is that you? Did you struggle with squaring and adding fractions in that last problem? Then it's *strongly* suggested you talk to your instructor about reviewing fractions. If you're having trouble with fractions now, it's only going to get worse.

Unfortunately, that's not all there is to completing the square. In all of the examples we've done so far, the leading coefficient has been 1. We haven't had anything like $2x^2 + 18x - 93$. Fortunately, we only need one more trick to handle cases like these. Let's just do an example. Say we have $2x^2 + 8x + 6$ and we want to rewrite it in the form $p(x + q)^2 + r$. Our first step will be to factor a 2 out of everything:

$$2x^2 + 8x + 6 = 2(x^2 + 4x + 3)$$

Now you see that part inside the parentheses? That's a quadratic, with a leading coefficient of 1. So we should be able to rewrite that as $(x + q)^2 + r$, using our old methods:

$$\begin{aligned} x^2 + 4x + 3 &= x^2 + 4x + 4 - 4 + 3 \\ &= (x + 2)^2 - 1 \end{aligned}$$

So the part inside the parentheses, $x^2 + 4x + 3$, is the same as $(x + 2)^2 - 1$. So let's go back to our original and substitute that in, and then distribute the leading 2 out again:

$$\begin{aligned} 2x^2 + 8x + 6 &= 2(x^2 + 4x + 3) \\ &= 2((x + 2)^2 - 1) \\ &= 2(x + 2)^2 - 2 \end{aligned}$$

And there we go, it's in the correct form. To recap: we factored out the leading coefficient, then completed the square on the "inside" part, and then multiplied that leading coefficient out again. Here's another example. Again, this one looks more complicated because it has fractions, but it's exactly the same procedure.

$$\begin{aligned} 3x^2 - 5x + 6 &= 3(x^2 - \frac{5}{3}x + 2) \\ &= 3(x^2 - \frac{5}{3}x + \frac{25}{36} - \frac{25}{36} + 2) \\ &= 3((x - \frac{5}{6})^2 - \frac{25}{36} + 2) \\ &= 3((x - \frac{5}{6})^2 + \frac{47}{36}) \\ &= 3(x - \frac{5}{6})^2 + \frac{47}{12} \end{aligned}$$

Your turn:

(7) $2x^2 - 16x + 8$

(8) $5x^2 + 10x + 15$

(9) $2x^2 + 5x + 1$

(10) $3x^2 + 3x + 7$

(11) $x^2 + x + 1$

(12) $2x^2 + x + 1$

(13) $10x^2 + 100x + 1000$

(14) $3x^2 + 4x + 5$